On the circular maps, bipartite maps and hypermaps which are self-equivalent with respect to reversing the colors of vertices

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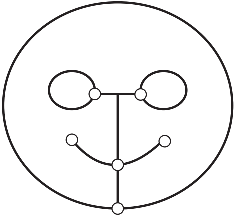
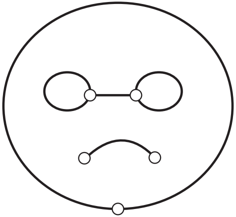
On the circular maps

A map (S, G) is a closed Riemann surface S with an embedded graph G such that $S \backslash G$ is homeomorphic to a disjoint union of open disks.



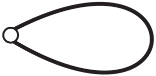
map



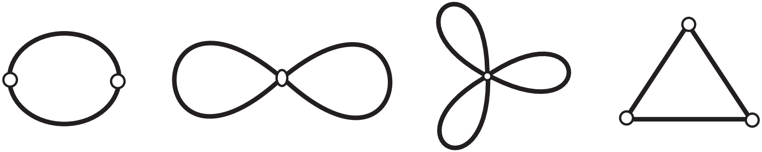
non-map

Two maps (S, G) and (S_1, G_1) are called equivalent whenever there exists an orientation-preserving homeomorphism $h : S \to S_1$ with $h(G) = G_1$.

Refer as an elementary circular map $(S_\circ, G_\circ)$ to a map on the sphere $S_\circ$ with one edge, one vertex, and two faces (inner and outer).



Refer as a circular map to a map covering an elementary circular map. In other words, (S, G) is a circular map if there exists a branched covering $f : (S, G) \to (S_\circ, G_\circ)$ ramified only over the centers of the faces and the vertex $G_\circ$ and such that $f(G) = G_\circ$.



Theorem 1. Denote by $C(n)$ the number of circular maps with n edges. Then

$$C(n) = \frac{1}{2n} \sum_{\substack{\ell | n \\ \ell m = n}} \left(s^{+}(m,0)\, \varphi_{m+1}(\ell) + \right.$$

$$+ Int(\frac{m}{2})\, (s(\frac{m}{2},0) - s^{+}(\frac{m}{2},0))\, \varphi^{odd}_{\frac{m}{2}+1}(\ell) +$$

$$\left. + \sum_{H=1}^{m} Int(\frac{m-H}{2})\, \frac{T(m,H)}{(m-1)!}\, \varphi_{\frac{m-H}{2}+1}(\ell) \right),$$

$\varphi_m(\ell)$ is the Jordan function, $\varphi^{odd}_{m+1}(\ell)$ is the odd Jordan function, $s(m,0)$, $s^{+}(m,0)$ calculated with two recurrence equations

$$s(n,0) = (2n+1)!! - \sum_{k=1}^{n} (2k-1)!!\, s(n-k,0), \;\; s(0,0) = 1,$$

and

$$s^{+}(n,0) = (n+1)! - \sum_{k=1}^{n} k!\, s^{+}(n-k,0), \;\; s^{+}(0,0) = 1,$$

respectively.

And $T(m,H)$ is calculated with recurrence formula:

$$T(m,H)=B(m,H)-\sum_{h=0}^{H}\sum_{i=1}^{m-1}\binom{m-1}{m-i}T(i,h)\,B(m-i,H-h)\,,$$

where

$$B(i,j)=i!\,\frac{i!}{1^j\,j!\,2^{\frac{i-j}{2}}\,\left(\frac{i-j}{2}\right)!}\,Int\left(\frac{i-j}{2}\right)\,,$$

$B(0,0)=1,\ T(0,0)=0,$

$$Int(x)=\begin{cases}1, \text{if } x\in Z \text{ and } x\geqslant 0,\\ \qquad 0, \qquad \text{else.}\end{cases}$$

Proposition 1. A planar map is a circular map if and only if it is an Euler map, i.e., its every vertex has even valence. This property fails for Riemann surfaces of greater genus.

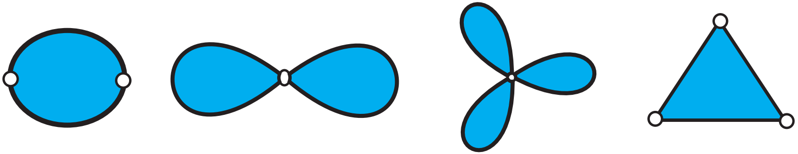
From this proposition it follows that the number of planar circular maps $C_0(n)$ coincides with the number of planar Euler maps with n edges. The last number was obtained by V. A. Liskovets in 2004.

n	$C(n)$	$C_0(n)$
1	1	1
2	2	2
3	5	4
4	16	12
5	56	34
6	333	154
7	2147	675
8	17456	3534
9	158022	18985
10	1604281	108070

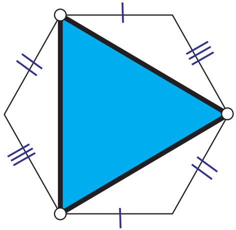
Table 1

Maps and coloring in two colors

Lemma 1. A map is circular if and only if *we can color* its faces in two colors so that each edge separates two different colors.



Circular map on a torus : 1 vertex, 3 edges, 2 faces

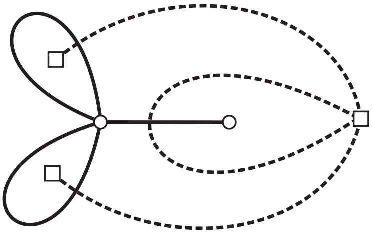


From Lemma 1 and Proposition 1 follows

Proposition 2. A planar map admits coloring of the faces in two colors if and only if its every vertex has even valence.

On the bipartite maps

Dual map. We put a new vertex inside each face of the original map (the "center" of the face). Then, for any edge of the original map, we draw a new edge which intersects it in its midpoint, and which connects the center of the two faces adjacent to this original edge. If these two faces coincide, the new edge thus obtained is a loop.



It is easy to see that the dual to the dual of a map is the original map itself. The dual map has the same number of edges as the original map.

A map is called bipartite map if its vertices *can be colored* in two colors (black and white) in such a way that every edge connects vertices of different colors.

Let us remark that a dual to a bipartite map is a circular map. As a result we obtain next theorem.

Theorem 2. The number $Bip(n)$ of bipartite maps with given number of edges n can be calculated by this formula

$$Bip(n) = \frac{1}{2n} \sum_{\substack{\ell | n \\ \ell m = n}} \Big(s^{+}(m,0)\, \varphi_{m+1}(\ell) +$$

$$+ Int(\frac{m}{2})\, (s(\frac{m}{2},0) - s^{+}(\frac{m}{2},0))\, \varphi^{odd}_{\frac{m}{2}+1}(\ell) +$$

$$+ \sum_{H=1}^{m} Int(\frac{m-H}{2})\, \frac{T(m,H)}{(m-1)!}\, \varphi_{\frac{m-H}{2}+1}(\ell) \Big),$$

$\varphi_m(\ell)$ is the Jordan function, $\varphi^{odd}_{m+1}(\ell)$ is the odd Jordan function.

On the hypermaps

A hypermap is a map whose vertices *are colored* in black and white in such a way that every edge connects vertices of different colors.

Two hypermaps (S, G) and (S_1, G_1) are called equivalent whenever there exists an orientation-preserving homeomorphism $h : S \to S_1$ with $h(G) = G_1$ and h taking black and white vertices of (S, G) to black and white vertices of (S_1, G_1), respectively .

Denote by $Hyp(n)$ the number of hypermaps with n edges up to hypermaps' equivalence.

From A. Mednykh and R. Nedela "Enumeration of unrooted hypermaps of a given genus" follows

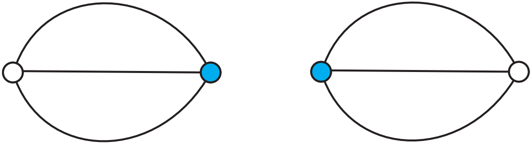
Proposition 3.
The number $Hyp(n)$ of hypermaps with given number of edges can be calculated by this formula:

$$Hyp(n) = \frac{1}{n} \sum_{\substack{l|n \\ lm=n}} s^{+}(m,0)\, \varphi_{m+1}(l)\,,$$

where $\varphi_{m+1}(l)$ is the Jordan function.

One can say that hypermap is self-equivalent with respect to reversing the colors of vertices, if given hypermap and hypermap which is obtained by reversing colors of vertices of given hypermap are equivalent.

Denote this number by $Shyp(n)$.



Note that the vertices of a bipartite map can be colored properly with black and white in two different ways unless it is self-equivalent with respect to reversing the colors of vertices.

Thus,

$$Shyp(n) = 2Bip(n) - Hyp(n).$$

Theorem 3. The number $Shyp(n)$ of hypermaps which are self-dual with respect to reversing the colors of vertices with given number of edges can be calculated by this formula:

$$Shyp(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \Big(Int(\frac{m}{2}) \, (s(\frac{m}{2}, 0) - s^{+}(\frac{m}{2}, 0)) \, \varphi^{odd}_{\frac{m}{2}+1}(\ell) +$$

$$+ \sum_{H=1}^{m} Int(\frac{m-H}{2}) \, \frac{T(m,H)}{(m-1)!} \, \varphi_{\frac{m-H}{2}+1}(\ell) \Big) \, ,$$

$\varphi_m(\ell)$ is the Jordan function, $\varphi^{odd}_{m+1}(\ell)$ is the odd Jordan function.

n	$Shyp(n)$	$Shyp_0(n)$
1	1	1
2	1	1
3	3	2
4	6	4
5	15	8
6	42	17
7	131	40
8	442	93
9	1551	224
10	5723	538

Table 2

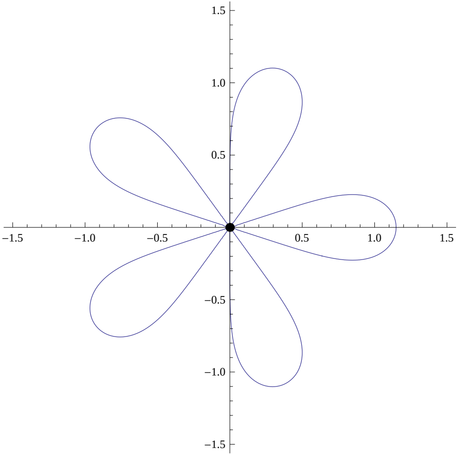
Note that the number $Shyp_0(n)$ of planar hypermaps which are self-dual with respect to reversing the colors of vertices was obtained by V. A. Liskovets in 2004.

On the Belyi functions for circular maps

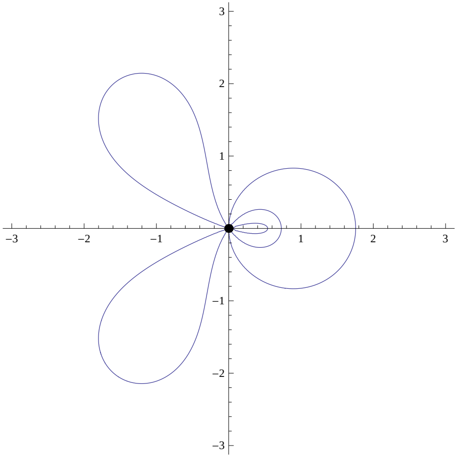
The list of Belyi functions for maps with the number of edges not exceeding four is contained in N. M. Adrianov, N. Ya. Amburg, V. A. Dremov, Yu. Yu. Kochetkov, E. M. Kreines, Yu. A. Levitskaya, V. F. Nasretdinova, and G. B. Shabat "Catalog of dessins d'enfants with no more than 4 edges". For planar circular maps with E edges they are represented as rational functions of degree $2E$.

We obtain Belyi functions for planar circular maps as rational functions of degree of E.

Map $(S_{\circ}, G_{78})$. Belyi function $\varphi_{0.78}(w) = w^5$.

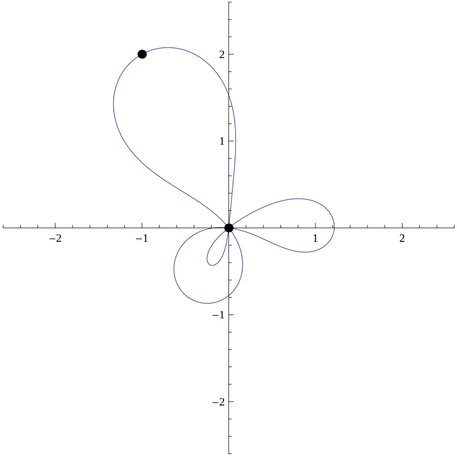


Map $(S_\circ, G_{81})$. Belyi function $\varphi_{0.81}(w) = \frac{4w^5}{(3-5w)^2}$.

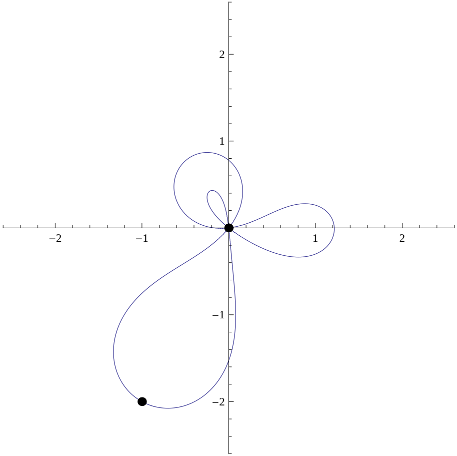


Map $(S_\circ, G_{100})$.

Belyi function $\varphi_{0.100}(w) = \frac{(-2i+1+w)w^4}{(-2iw+1+w)}$.



Map $(S_\circ, G_{101})$.
Belyi function $\varphi_{0.101}(w) = \frac{(2i+1+w)w^4}{(2iw+1+w)}$.



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Thank you for your attention!